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SOME ELEMENTARY THEOREMS CONCERNING THE STEADY FLOW OF ELECTRICITY IN SOLID CONDUCTORS

By B. O. Peirce

Although many familiar mathematical truths were first discovered by the study of physical phenomena, and many propositions, the analytical proofs of which are somewhat long and involved, are nearly self-evident when regarded from the point of view of their physical consequences, it is, nevertheless, true that the analytical treatment of a theorem almost always throws new light upon it and shows clearly what its limitations are, if not in what respects it may be stated more generally. It is sometimes true also that reasoning which seems satisfactory enough to a person who has had long experience in making physical measurements, is not so convincing to a student of Mathematics whose knowledge of Physics is largely theoretical. For these reasons it has seemed to me worth while to discuss briefly, from their mathematical side, a few simple theorems concerning the steady flow of electricity in massive conductors; some of these, as being self-evident, are stated without proof in most treatises on the subject; others are rarely, if ever, mentioned.

It is usual to assume* that in an isotropic conductor, K, which is carrying a steady current of electricity, the current vector q has everywhere the direction of the electrostatic field, F, due to all the free electricity in existence, and that the strength of the current at any point is equal to λ times the strength of F at the point, where λ is an intrinsically positive scalar function of the space coordinates called the *specific conductivity* of K. The value of λ is determined by the physical nature of K and by the temperature; the specific conductivity of a homogeneous conductor has the same value at all points of the conductor. The fact that after a steady current has been established in a conductor kept at

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* Riemann-Weber: *Die partiellen Differentialgleichungen der mathematischen Physik*, §§ 162, 163.

Webster: *The theory of electricity and magnetism*, chap. VIII, part II.
Maxwell: *A treatise on electricity and magnetism*, vol. 1, part II, chap. VIII.
Gray: *Absolute measurements in electricity and magnetism*, vol. 1, pp. 161–164.
Mascart et Joubert: *Leçons sur l'électricité et le magnétisme*, §§ 198–242.
Curry: *Theory of electricity and magnetism*, § XXII.

a uniform temperature, there is no growing accumulation of free electricity in any portion of the conductor leads to the equation of continuity

$$\text{Divergence } q = 0,$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

where u, v, w are the components of the current, parallel to the coordinate axes.

If V is the electrostatic potential function and if X, Y, Z are the components of the electrostatic field,

$$u = \lambda \cdot X = -\lambda \cdot \frac{\partial V}{\partial x}, \qquad v = \lambda \cdot Y = -\lambda \cdot \frac{\partial V}{\partial y}, \qquad w = \lambda \cdot Z = -\lambda \cdot \frac{\partial V}{\partial z};$$

and the equation of continuity expresses the fact that the divergence of λ times the gradient vector of V is zero; that is

$$\frac{\partial}{\partial x}\left(\lambda \cdot \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \cdot \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \cdot \frac{\partial V}{\partial z}\right) = 0. \tag{2}$$

We shall find it convenient to represent the first member of (2) by the symbol $\Omega(\lambda, V)$.

At the common surface S_{12} of two conductors K_1, K_2 which are carrying a steady current of electricity, the continuity of flow leads to the equation

$$\lambda_1 \cdot \frac{\partial V}{\partial n_1} + \lambda_2 \cdot \frac{\partial V}{\partial n_2} = 0, \tag{3}$$

where λ_1, λ_2 are the specific conductivities of the conductors, and n_1, n_2 represent the directions of the normals to S_{12} at any point on it drawn into K_1 and K_2 respectively. S_{12} may or may not be an equipotential surface; in any case, according to Volta's theory, if the two conductors are made of different materials, there is, in general, a discontinuity of electrostatic potential at their common surface of contact. The magnitude (E_{12}) of this discontinuity, that is, the sudden increase in the value of the potential function at a moving point P as P leaves K_1 and enters K_2, is the same for every point of S_{12} and depends upon the materials of which the conductors are made and upon their temperatures.

The surface of separation (S) of a dielectric and a conductor which is carrying a steady current of electricity has in general an electrostatic charge: at any point of such a surface the derivative of the potential function taken in the direction of the normal pointing into the conductor is zero, but the derivative

of the potential function outside the conductor, taken in the direction of the normal pointing into the dielectric, is not usually zero. The equipotential surfaces within the conductor meet S at right angles; the equipotential surfaces without the conductor meet S in general obliquely.

If a conductor which contains within itself no source or sink of electricity is carrying a steady current, current filaments enter the conductor through some portion or portions of its surface called the *anode* or the *anodes*, and current filaments leave the conductor through some other portion or portions of the surface called the *kathode* or the *kathodes*. The derivative of the potential function in the direction of the interior normal to the surface of the conductor is negative at every point of an anode, positive at every point of a kathode.

If every point of a portion (A) of the surface of a simple conductor be kept at the same potential (V_A), and every point of another portion (B) at another potential (V_B) while the rest of the surface abuts on an insulating medium, A and B are *equipotential electrodes*; and if under these circumstances a steady current of strength C passes from A to B the resistance r of the conductor is defined to be $(V_A - V_B)/C$.

An amount of heat equal to $(V_A - V_B)C$ or C^2r is set free in the conductor every second by the current. If n indicates a normal to the surface of the conductor pointing inwards, the strength of the current, C, is equal to the surface integral of $-\lambda \cdot \partial V/\partial n$, taken over A, or to the surface integral of $+\lambda \cdot \partial V/\partial n$, taken over B; and the Joulean heat is equal to

$$-\int\int V \cdot \lambda \cdot \frac{\partial V}{\partial n} \cdot dS, \tag{4}$$

taken over A *and* B, or, since this will give the same result, over the entire surface of the conductor. Everywhere within the conductor $\Omega(\lambda, V)$ vanishes and a simple application of Green's Theorem gives for the Joulean heat the volume integral, taken through the conductor, of λ times the square of the strength of the electrostatic field, or

$$\int\int\int \lambda \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right\} d\tau; \tag{5}$$

this integral extended over the space occupied by a conductor K may be denoted by the symbol

$$J(\lambda, V)_K.$$

If the first of an open chain of conductors has an equipotential electrode

A kept at potential V_A, and the last an equipotential electrode B kept at the lower potential V_B, if, moreover, the algebraic sum of the discontinuities of the potential function at the common surfaces of consecutive conductors (counting a sudden *increase* on the way from A to B as positive) is E, the Joulean heat set free per second within the conductors which form the chain is

$$(V_A - V_B + E)C$$

and this is equal to $J(\lambda, V)$ extended throughout the space occupied by all the conductors. This expression takes no account of the thermal phenomena at the junctions.

If at every point of a portion of the boundary of a region R in space, the value of a function V is given, and at every point of the rest of the boundary the value of the interior-normal derivative of V, and if $\Omega(\lambda, V)$ vanishes within R where λ is a given positive point function which, with its space derivatives of the first order, is finite within R, then V, if it exists, is unique. A problem concerning the steady flow of electricity in a conductor is considered solved, therefore, when a function V has been somehow discovered which is found on trial to satisfy the prescribed conditions for the potential function at the surface of the conductor, and to obey within the conductor the requirement $\Omega(\lambda, V) = 0$.

If over a portion of the boundary S of a given space region R, in which the functions V and V' with their derivatives of the first and second orders are finite, these functions have equal values at every point, while over the remainder of S their normal derivatives vanish; and if within S, $\Omega(\lambda, V) = 0$ and $\Omega'(\lambda, V') \neq 0$, where λ is a positive point function finite with its first derivatives in R: then $J(\lambda, V')_R$ is greater than $J(\lambda, V)_R$.

If there is no flux across a given surface within or at the boundary of a conductor which is carrying a steady current, the surface is called a *current surface* or a *surface of flow*.

A conductor is said to be *given* when its dimensions and its specific conductivity are given; the surface conditions of a conductor which is carrying a steady current are said to be the same at two different times when the electrodes and the values of the potential function at every point of the electrodes are the same. Unless the contrary is stated, the electrodes of a conductor are usually assumed to be equipotential.

The surfaces of equal conductivity ($\lambda = k$) in a conductor may coincide with the equipotential surfaces; if they happen to do so in any given case, a

change in the dimensions or position of an electrode will generally destroy the coincidence.

The flow within an open chain formed of given conductors with given discontinuities of potential at the given surfaces of contact of the successive conductors in the chain, is evidently determined when the value of either the potential function or its normal derivative is given at every point of the free surface of the chain.

If the problem of electrical flow in a given conductor under given surface conditions has been solved, it is possible to write down immediately the value which the potential function in the conductor would have had under the same superficial conditions if the specific conductivity had been changed in certain ways; that is, the solution of one problem in Electrokinematics gives the solutions of certain other related problems. For example: the potential function in a conductor under given surface conditions has the same value as it would have under the same superficial conditions if the specific conductivity, $\lambda = \phi(x,y,z)$, were everywhere increased to m times its old value; the heat developed in the conductor per second would be increased m-fold by the change of conductivity and the resistance of the whole conductor would be decreased in the same ratio.

Given the potential function (V) in any homogeneous conductor which is carrying a steady current under fixed surface conditions, the specific conductivity can be changed in many different ways without changing the value of the potential function; indeed λ may have any value which will make $\Omega(\lambda, V)$ vanish within the conductor. Since V is harmonic, the equation $\Omega(\lambda, V) = 0$ can be written in the form

$$\frac{\partial V}{\partial x} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial V}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \frac{\partial V}{\partial z} \cdot \frac{\partial \lambda}{\partial z} = 0. \qquad (6)$$

So that λ, if not constant, may be equal to any point function which yields level surfaces perpendicular to the equipotential surfaces.

If the potential function V' in a non-homogeneous conductor is to have the same level surfaces as the potential function V of a homogeneous conductor of the same dimensions under the same surface conditions, we may write $V' = f(V)$ and determine λ from the equation $\Omega(\lambda, V') = 0$, or

$$\lambda \cdot f'(V) \cdot \nabla^2(V) + f''(V) \cdot \left(\frac{\partial V}{dx} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial V}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \frac{\partial V}{\partial z} \cdot \frac{\partial \lambda}{\partial z} \right) + \lambda \cdot h_V^2 \cdot f''(V) = 0, \qquad (7)$$

where h_V is the gradient of V or the strength of the field within the homogeneous conductor. Now the normal derivative* of the logarithm of λ, taken with respect to V, *i. e.* $[D_V(\log\lambda)]$, is equal to

$$\left(\frac{\partial V}{\partial x}\cdot\frac{\partial\lambda}{\partial x}+\frac{\partial V}{\partial y}\cdot\frac{\partial\lambda}{\partial y}+\frac{\partial V}{\partial z}\cdot\frac{\partial\lambda}{\partial z}\right)\Big/\left(\lambda\cdot h_V^2\right), \tag{8}$$

and V is harmonic, so that (7) reduces to the form

$$\frac{f''(V)}{f'(V)}=-[D_V(\log\lambda)]. \tag{9}$$

If the function V be used to define one of a set of orthogonal curvilinear coordinates (U, V, W), we learn from integrating equation (9) that $\lambda\cdot f'(V)$ must be independent of V and is, therefore, either an absolute constant or a function of U and W. The surface conditions require that V and $f(V)$ shall be equal at each electrode: otherwise f is wholly arbitrary. If, therefore, (to consider a simple application of what precedes) we have the solution of a problem of steady flow within a homogeneous conductor, then the potential function, under the same superficial conditions, in a stratified conductor which has the same dimensions as the other but a specific conductivity represented by any desired function (ϕ) of the old potential function (V), is of the form $M\cdot F(V)+N$, where $F(V)$ is any integral of the reciprocal of $\phi(V)$ and M, N are constants so chosen as to satisfy the surface conditions. Although the reciprocal of $\phi(V)$ must be integrable, ϕ itself need not be everywhere continuous.

If a conductor which carries a steady current is not homogeneous the potential function (V) within it is not generally harmonic, indeed it is usually true that no harmonic function exists which has the same level surfaces as V. If in such a case the dimensions of the conductor are given as well as the potential function V within it, so that the superficial conditions are determined, the specific conductivity is evidently not determined. The function λ need only satisfy the equation $\Omega(\lambda, V)=0$, or

$$\lambda\cdot\nabla^2(V)+\lambda\cdot h_V^2[D_V(\log\lambda)]=0, \tag{10}$$

or

$$[D_V(\log\lambda)]=-\frac{\nabla^2 V}{h_V^2} \tag{11}$$

* Peirce: *The Newtonian potential function*, p. 116; *A short table of integrals*, p. 106.

Unless Lamé's condition* for isothermal parameters happens to be satisfied, the second member of (11) usually involves U, V, and W.

If V is not harmonic, but if the λ and V surfaces coincide, λ is a function of V and the condition $\Omega(\lambda, V) = 0$ shows that the ratio of $\nabla^2(V)$ to h_V^2 is expressible as a function of V; it is possible, therefore, to find a harmonic function which is expressible in terms of V alone.

We may write equation (2) in the form $\frac{\nabla^2(V)}{h_V^2} = -\frac{h_\lambda \cdot \cos(\lambda, V)}{\lambda \cdot h_V}$ and assert that if the λ and V surfaces are neither coincident nor orthogonal, V cannot be harmonic.

If the steady flow of electricity in a conductor is defined by a potential function the level surfaces of which are neither coincident with nor orthogonal to the surfaces of constant λ, it is not possible to change the specific conductivity to a new value expressible in terms of the old one — except by altering its value at every point in a constant ratio — if under the same surface conditions, the potential function is to be unchanged. This follows immediately from a comparison of the equations $\Omega(L, V) = 0$ and $\Omega(\lambda, V) = 0$, in which L is supposed to be a function (ϕ) of λ; for if we multiply the second of these by $\phi'(\lambda)$ and subtract it from the first, it appears that both can be satisfied only if

$$\nabla^2(V)[\phi(\lambda) - \lambda \cdot \phi'(\lambda)] = 0, \tag{12}$$

and, since V is not harmonic, the quantity in square brackets must vanish. If, however, the new value of the specific conductivity be not necessarily a function of the old value alone, we have according to (11),

$$[D_V \log L] = [D_V \log \lambda] \text{ or } L = \lambda \cdot F(U, W).$$

If in a conductor which is carrying a steady current, the equipotential surfaces and the surfaces of constant conductivity coincide, so that $\lambda = \phi(V)$ where V is the potential function, then

$$\Omega[\phi(V), V] = 0: \tag{13}$$

if with the same surface conditions the conductivity be changed to λ', a function of the old λ alone, $[\chi(\lambda) = \chi\phi(V) = \psi(V)]$, then the new potential function, V', will be a function of the old one $[V' = f(V)]$, and we shall have

$$\Omega[\psi(V), f(V)] = 0. \tag{14}$$

* Lamé: *Leçons sur les coordonnées curvilignes*, p. 31; *Leçons sur les fonctions inverses*, p. 5. Somoff-Ziwet: *Theoretische Mechanik*, vol. 1, p. 113, and p. 128.

Equations (13) and (14) when expanded yield the relation

$$\frac{f''(V)}{f'(V)} + \frac{\psi'(V)}{\psi(V)} = \frac{\phi'(V)}{\phi(V)} = -\frac{\nabla^2(V)}{h^2_V}; \tag{15}$$

whence

$$\phi = e^{-\int \frac{\nabla^2(V)dV}{h^2}}, \tag{16}$$

$$\log[f''(V)\cdot\psi(V)] = \log[K\cdot\phi(V)], \tag{17}$$

$$V' = f(V) = K\int\frac{\phi(V)}{\psi(V)}dV + L. \tag{18}$$

In equation (18) which gives the new potential function in terms of the old one, K and L are constants which, after a value has been assigned to the indefinite integral, are to be chosen so as to satisfy the surface conditions.

If V is the potential function in a conductor which has two equipotential electrodes A and B, when these are kept at potentials V_A and V_B, respectively, the value of the potential function when the electrodes are kept at potentials V'_A, V'_B is the linear function of V given by the equation

$$V' = \frac{V'_A - V'_B}{V_A - V_B}\cdot V + \frac{V'_B V_A - V'_A V_B}{V_A - V_B}, \tag{19}$$

for $\Omega(\lambda, V') = 0$, and V' satisfies the surface conditions.

Let $V = V_1$, $V = V_2$ be the equations of two equipotential surfaces S_1, S_2 which cut completely across a conductor which is carrying a steady current of electricity in such a manner that the equipotential surfaces are surfaces of constant conductivity so that $\lambda = \phi(V)$. If then the conductivity of the portion of the conductor between S_1 and S_2 be changed so as to become $\lambda' = \psi(V)$, while the surface conditions are unchanged, we may consider the new conductor as a chain of three separate conductors, each of which has two equipotential electrodes. We must assume in general that there is a discontinuity of conductivity at S_1 and at S_2, and that accompanying this (if not in consequence of it) there are discontinuities $(E_1, -E_2)$ of the potential function at these surfaces. The surfaces A and S_1 which in the original conductor were equipotential are still equipotential in the new one though the value of the potential function on S_1 has been changed; it is evident therefore from the statement in the last paragraph that the new potential function in this portion of the conductor must be a linear function, $MV + N$, of the old potential function. Similar reasoning

shows that the new potential function in the portion of the conductor which lies between S_2 and B must be of the form $QV + T$, and the potential function in the conductor between S_1 and S_2 of the form $K \cdot f(V) + L$, where $f(V)$ satisfies equation (18).

The six constants M, N, K, L, Q, T are to be determined from the equations

$$M V_1 + N + E_1 = K \cdot f(V_1) + L, \qquad K_1 \cdot f(V_2) + L = Q V_2 + T + E_2,$$
$$M V_A + N = V_A, \qquad Q V_B + T = V_B,$$
$$M \lambda_1 = K \cdot \lambda'_1 [f'(V)]_{V_1}, \qquad Q \cdot \lambda_2 = K \cdot \lambda'_2 [f'(V)]_{V_2}.$$

If in accordance with equation (17) we put $f'(V) = \lambda/\lambda'$, it appears that

$$M = K = Q = \frac{V_A - V_B + E_1 - E_2}{\Delta},$$

$$N = \frac{V_A [f(V_1) - f(V_2) + V_2 - V_1 + E_2 - E_1]}{\Delta},$$

$$T = \frac{V_B [f(V_1) - f(V_2) + V_2 - V_1 + E_2 - E_1]}{\Delta},$$

$$L = \frac{(V_B + E_2) [f(V_1) + V_A - V_1] - (V_A + E_1) [f(V_2) + V_B - V_2]}{\Delta},$$

where
$$\Delta = f(V_1) - f(V_2) + V_2 - V_1 + V_A - V_B. \qquad (21)$$

If V is the potential function in a homogeneous conductor of conductivity λ, which has two equipotential electrodes A, B, and if $V = V_0$ is the equation of an equipotential surface S_0 which separates the conductor into two parts, then if the portion of the conductor between S_0 and B changes its conductivity to the uniform value λ'_1 the new potential function, V', will have the value $MV + N$ between A and S_0 and the value $QV + T$ between S_0 and B. We may assume that there will be a discontinuity of potential, E, at S_0. Since $MV_A + N = V_A$, $QV_B + T = V_B$, $QV_0 + T = MV_0 + N + E$, $\lambda M = \lambda' Q$, we have

$$M = \frac{\lambda'}{\lambda} \cdot Q = \frac{\lambda'(V_A - V_B + E)}{\lambda V_0 - \lambda' V_0 - \lambda V_B + \lambda' V_A},$$

$$N = \frac{V_A [V_B (\lambda' - \lambda) + V_0 (\lambda - \lambda') - \lambda' E]}{\lambda V_0 - \lambda' V_0 - \lambda V_B + \lambda' V_A},$$

$$T = \frac{V_B\,[\,V_0\,(\lambda - \lambda') + V_A\,(\lambda' - \lambda) - \lambda E]}{\lambda V_0 - \lambda' V_0 - \lambda V_B + \lambda' V_A} \tag{21}$$

If one of two conductors (K_1, K_2) which abut at a common surface S_0, has an equipotential electrode A kept at potential V_A and the other an equipotential electrode B kept at potential V_B, and if V is the potential function within the compound conductor when the sudden increase in V in crossing S_0 from K_1 to K_2 is E: then the potential function, V', has in the two conductors the values

$$\frac{(E'' + V_A - V_B)\,V + V_A(E - E'')}{E + V_A - V_B},$$

$$\frac{(E'' + V_A - V_B)\,V + V_B(E - E'')}{E + V_A - V_B}, \tag{22}$$

when E'' is the discontinuity of potential at S_0 and the other conditions are the same.

If a conductor R has three equipotential electrodes A, B, C kept at three different potentials, we cannot speak of the resistance* of the conductor, though the flow of electricity within it is determined when its conductivity is known, and the development of heat in it per second is perfectly definite. If we have found the values V', V'' of the potential function in R for two distinct sets of surface conditions:

(1) when A, B, C are kept at potentials V'_A, V'_B, V'_C respectively, and

(2) when they are kept at potentials V''_A, V''_B, V''_C:

the potential function when the electrodes are kept at potentials V_A, V_B, V_C is usually of the form

$$V = LV' + MV'' + N; \tag{23}$$

since $\Omega(\lambda, V) = 0$ within R, and at every point of its surface outside of the electrodes $\partial V/\partial n = 0$: if, therefore, L, M, N can be made to satisfy the equations

$$\begin{aligned} LV'_A + MV''_A + N &= V_A, \\ LV'_B + MV''_B + N &= V_B, \\ LV'_C + MV''_C + N &= V_C, \end{aligned} \tag{24}$$

* Maxwell: *A treatise on electricity and magnetism*, §306.

the statement just made is justified. The procedure fails, however, if

$$\begin{vmatrix} V'_A & V''_A & 1 \\ V'_B & V''_B & 1 \\ V'_C & V''_C & 1 \end{vmatrix} = 0, \tag{25}$$

for then

$$V'' = \frac{(V''_A - V''_B)V' + V'_A V''_B - V''_A V'_B}{V'_A - V'_B}, \tag{26}$$

the equipotential surfaces are the same in both solutions, and the cases are not really distinct.

Since in general $\Omega(\lambda, V') = 0$, and $\Omega(\lambda, V'') = 0$ within R, we may learn from Green's Theorem that

$$\int\int \lambda \cdot V' \cdot \frac{\partial V''}{\partial n} \cdot dS = \int\int \lambda \cdot V'' \cdot \frac{\partial V'}{\partial n} \cdot dS, \tag{27}$$

where the integrations are to be extended over the surface of R. If then I'_A, I'_B, I'_C represent the fluxes inward across the electrodes in the first case and I''_A, I''_B, I''_C the corresponding quantities in the second case,

$$V'_A \cdot I''_A + V'_B \cdot I''_B + V'_C \cdot I''_C = V''_A \cdot I'_A + V''_B \cdot I'_B + V''_C \cdot I'_C, \tag{28}$$

where

$$I'_A + I'_B + I'_C = I''_A + I''_B + I''_C = 0. \tag{29}$$

If A, B, C are given equipotential portions of the free surface of an open chain of conductors of given conductivities with given discontinuities of potential at the surfaces of contact of successive members of the chain; and if V', V'', V are the potential functions within the chain, when A, B, C are kept at potentials V'_A, V'_B, V'_C; V''_A, V''_B, V''_C; V_A, V_B, V_C, respectively; then V is a linear function of V' and V'' in every conductor, if the solutions are distinct. This treatment of the flow within conductors which have two and three electrodes can be easily extended to the flow in a chain of m given conductors with p given equipotential electrodes.

A change in the specific conductivity of a portion of a conductor which has given electrodes, is usually accompanied by a corresponding change in the resistance of the conductor as a whole and by a change in the distribution of the current in all parts of the conductor, though two or three exceptional cases

have been considered in the preceding pages. If V is the potential function in a conductor which has two fixed equipotential electrodes A, B, kept at potentials V_A, V_B, and if the specific conductivity of a portion P of the conductor be changed in any manner, provided that its value is increased at every point of P, the amount of heat developed per second in the whole conductor will be increased and its resistance* decreased. The change in the conductivity of P will either leave the potential function V unchanged or it will change V to V'. In the first case, h_V, the strength of the field, will be unchanged, but the positive quantity λh_V^2 (the integral of which taken through the conductor measures the rate at which heat is developed in it) is larger in P than it was before and in the remainder of the conductor has its old value, so that the integral will have a greater value than before. If the new potential function, as will generally be the case, differs from the old one, denote their difference by u, so that $V' = V + u$. At every point of either electrode $V = V'$, so that $u = 0$: at every other point of the boundary of the conductor $\partial V/\partial n = 0$, $\partial V'/\partial n = 0$, so that $\partial u/\partial n = 0$: within the conductor $\Omega(\lambda, V) = 0$, $\Omega(\lambda, V') = 0$, so that $\Omega(\lambda, u) = 0$: we have, therefore, [see equation (5)]

$$J(\lambda, V') = J(\lambda, V) + J(\lambda, u) + 2\iiint \lambda \left(\frac{\partial V}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial V}{\partial y} \cdot \frac{\partial u}{\partial y} + \frac{\partial V}{\partial z} \cdot \frac{\partial u}{\partial z} \right) d\tau \qquad (30)$$

$$= J(\lambda, V) + J(\lambda, u) - 2\iint \lambda \cdot u \cdot \frac{\partial V}{\partial n} \cdot dS - 2\iiint u \cdot \Omega(\lambda, V) d\tau \qquad (31)$$

The integrand of the surface integral in (31) vanishes at every point of the boundary of the conductor, for at the electrodes $u = 0$, and at the surface of separation of the conductor and the surrounding dielectric the normal derivative of V taken inwards must vanish; the integrand of the volume integral vanishes also at every point of the conductor, so that

$$J(\lambda, V') = J(\lambda, V) + J(\lambda, u). \qquad (32)$$

* Rayleigh: On the theory of resonance, *Phil. Trans.* vol. 161, 1871; *Scientific papers*, vol. 1, p. 51.

Webster: *The theory of electricity and magnetism*, § 175.

Maxwell: *A treatise on electricity and magnetism*, §§ 306, 307.

Now $J(\lambda,u)$ is not zero, for u is not a constant: it is not negative, for its integrand cannot be negative; therefore $J(\lambda,u)$ is positive and

$$J(\lambda, V'') > J(\lambda, V') ;$$

but

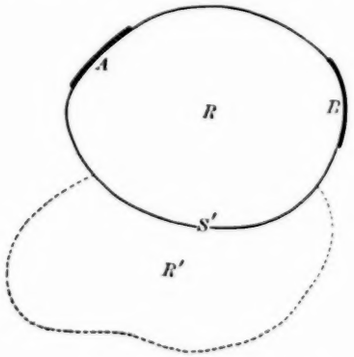
$$J(\lambda', V'') > J(\lambda, V'') ;$$

therefore, *a fortiori*,

$$J(\lambda', V'') > J(\lambda, V'). \tag{33}$$

The heat developed in the conductor, per second, by a steady current must be equal to the product of the constant $(V_A - V_B)$ and the strength of the current; in the present case, therefore, the current must be greater when the conductivity of P' is increased and the resistance of the whole conductor must be less.

If a conductor, R, which has two equipotential electrodes, A and B, and is carrying a steady current be increased in bulk by bringing up to its side another conductor R' which has no electrodes of its own but is in electrical contact with R over a portion S' of its surface outside of A and B, then the composite conductor will always have a smaller resistance than R has alone. Since parallel tangential derivatives of the potential function at points close



together but on opposite sides of S' must have equal limits, either S' must have been made equipotential by the coming of R' or there must be a potential gradient in R' near S' and hence a flow of electricity in R' which has no electrodes and must receive any current which it has through current filaments which cross S'. In either case S' cannot continue to be a surface of flow and the po-

tential function in R cannot be the same as it was before R' was brought up. Let V and V'' be the old and the new potential functions and write $u = V'' - V$; then

$$J(\lambda, V'')_R = J(\lambda, V)_R + J(\lambda, u)_R - 2\int\int \lambda \cdot u \cdot \frac{\partial V}{\partial n} \cdot dS$$

$$- 2\int\int\int u \cdot \Omega(\lambda, V) d\tau, \qquad (34)$$

where the volume integral is to be taken through R and the double integral over its surface. Since the integrands vanish, we have

$$J(\lambda, V'')_R > J(\lambda, V)_R. \qquad (35)$$

so that the development of heat *in R alone* is greater after the bringing up of R' than before. Heat is also developed in R' and the current in the composite conductor is greater than the original current in R.

If in a conductor R which is carrying a steady current of electricity under given superficial conditions a surface S' not previously equipotential be suddenly forced to become so, by making the specific conductivity at all points on and very near to S' infinite, while the rest of R is unchanged, the resistance of the conductor will be decreased. In this theorem S' may represent a closed surface which shuts in a finite part of the conductor and excludes a part, it may be an open surface which cuts completely across the conductor, or it may be practically a portion of an open surface bounded by a curve wholly within or partly on the surface of the conductor. In any case if S' was not previously equipotential the new potential function, V'', must differ from the old one, and if we enclose S' between two parallel surfaces close to it, write $V'' - V = u$, and compute the values of $J(\lambda, V'')$ and $J(\lambda, V)$ for the space within R modified in this way, we shall get an equation which is formally the same as (35).

If an "infinitely thin, insulating membrane" were suddenly inserted into R so as to coincide with an open or closed surface S'', wholly within R, across which there had previously been a flux, the resistance of the conductor would be increased. If V is the actual potential function and V''' the potential function in the supposed case, we may write $V - V''' = u$, shut in S'' between two parallel surfaces S''_1, S''_2 drawn very close to it, find the values of $J(\lambda, V)$, $J(\lambda, V''')$

for the region within the conductor thus modified, and suppose S_1'' and S_2'' to approach each other indefinitely. We learn that

$$J(\lambda, V) = J(\lambda, V'') + J(\lambda, u) - 2\int\int \lambda \cdot u \cdot \frac{\partial V''}{\partial n} \cdot dS$$
$$- 2\int\int \lambda \cdot u \cdot \frac{\partial V''}{\partial n} \cdot dS_1'' - 2\int\int \lambda \cdot u \cdot \frac{\partial V''}{\partial n} \cdot dS_2'', \qquad (36)$$

where the first surface integral is to be extended over the outer boundary of the conductor. Since the derivative of V'' in a direction normal to S'' is equal to zero at every point on either side of this surface, the integrals vanish and $J(\lambda, V) = J(\lambda, V'') + J(\lambda, u)$, so that

$$J(\lambda, V'') < J(\lambda, V). \qquad (37)$$

If an interior portion, R_0, of a conductor R which is carrying a steady current of electricity under given surface conditions could be displaced by matter having at every point the same specific conductivity as before but of such a nature that at every point of S_0, the boundary of R_0, the potential function just within this surface should be greater than the potential function just without by a given constant amount E, then the resistance of the conductor as a whole would not be changed. Suppose the old and the new potential functions to be V and V_0, shut S_0 in between two near parallel surfaces S_0', S_0'', find the value of $J(\lambda, V_0 - V)$ extended through all the conductor which is not between S_0', S_0'', and determine the limit of this value as these surfaces approach each other indefinitely. If $u = V_0 - V$, u is zero at every point of either electrode and $\partial u/\partial n$ is zero at every point of the boundary of the conductor outside of the electrodes. The value of u at any point just inside S_0 differs from the value of u at a neighboring point just outside S_0 by E. It is evident from equation (3) that the normal derivatives of V and V_0 are continuous at S_0.

$$J(\lambda, u) = -\int\int \lambda \cdot u \cdot \frac{\partial u}{\partial n} \cdot dS - \int\int \lambda \cdot u \cdot \frac{\partial u}{\partial n} \cdot dS_0' - \int\int \lambda \cdot u \cdot \frac{\partial u}{\partial n} \cdot dS_0'', \qquad (38)$$

where the first integral is to be extended over the outer boundary of the conductor, and n indicates the directions of a normal pointing into the field of the volume integral represented by the first member. The integrand of

the first integral vanishes: the limit of the sum of the other two integrals is

$$-E\int\int \lambda \cdot \frac{\partial u}{\partial n} \cdot dS_0. \tag{39}$$

where n points into the region bounded by S_0. By a simple application of Green's Theorem, we have, therefore,

$$J(\lambda, u)_R = E\int\int\int \Omega(\lambda, u) d\tau, \tag{40}$$

where the volume integral is to be extended through the space bounded by S_0: but in this region $\Omega(\lambda,u) = 0$, so that the integral $J(\lambda,u)_R$ is equal to zero, and u must be constant. It is evident that since $u = 0$ at an electrode, it has the same value everywhere inside the conductor outside S_0: within S_0, it has everywhere the value E. At every point within R, the field is the same in direction and strength after the change and before it, so that

$$J(\lambda, V) = J(\lambda, V_0). \tag{41}$$

Given a conductor of any shape which has two equipotential electrodes, A and B, kept at potentials V_A, V_B and which is carrying a steady current of electricity, it is easy to see that if one electrode (say A) be enlarged so as to occupy an additional portion A' of the surface of the conductor, the resistance of the conductor will be decreased. If V is the old potential function and V' the new one, $\partial V/\partial n$ is zero on A', V' is equal to V_A on A'. If $u = V' - V$,

$$J(\lambda, V') = J(\lambda, V) + J(\lambda,u) - 2\int\int \lambda \cdot u \frac{\partial V}{\partial n} dS - 2\int\int\int u \cdot \Omega(\lambda, V) d\tau \tag{42}$$

$$= J(\lambda, V) + J(\lambda, u). \tag{43}$$

$J(\lambda,u)$ is positive since u is not constant, therefore

$$J(\lambda, V') \text{ is greater than } J(\lambda, V).$$

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ON A LINEAR TRANSFORMATION, AND SOME SYSTEMS OF HYPOCYCLOIDS*

By R. E. Allardice

In the discussion of the envelope of the asymptotes of a system of similar conics through three fixed points,† I was led to the following equation in tangential coordinates:

$$\Sigma \sin^2 C(uv^2 \cos A + u^2 v \cos B) - 2(1 + \cos A \cos B \cos C)uvw$$
$$+ m(v \sin C - w \sin B)(w \sin A - u \sin C)(u \sin B - v \sin A) = 0, \qquad (1)$$

where m is a constant proportional to the cotangent of the angle between the asymptotes. It is easy to show that the curve represented by this equation has the straight line at infinity for a double tangent, the circular points being the points of contact. Hence the curve is of the fourth order and has three cusps; it is therefore a three-cusped hypocycloid, for this is the only three-cusped quartic that has the straight line at infinity for a double tangent, with the circular points as the points of contact.

The problem naturally suggested itself of transforming this equation to the form

$$(u + v + w)^3 - 27uvw = 0, \qquad (2)$$

where the triangle of reference is an equilateral triangle. This was accomplished by making use of the known forms of the tangential equations of the circular points at infinity, and using for the third equation that of the centre of the hypocycloid.

Thus we have the equations

$$\begin{aligned} ue^{\alpha i} - ve^{(C+\alpha)i} - we^{-(B-\alpha)i} &= x + \omega y + \omega^2 z \equiv \lambda \text{ (say)}, \\ ue^{-\alpha i} - ve^{-(C+\alpha)i} - we^{(B-\alpha)i} &= x + \omega^2 y + \omega z \equiv \mu, \\ p_1 u + p_2 v + p_3 w &= x + y + z \equiv \nu, \end{aligned}$$

where α, p_1, p_2, p_3 are constants to be determined, and ω is an imaginary cube root of unity. The equation (1) may be written in the form

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* Since some of the results in this paper are identical with some of the theorems given by H. A. Converse: "On a system of hypocycloids of class three inscribed to a given 3-line, and some curves connected with it" (Annals of Mathematics, second series, vol. 5, pp. 105–139, April, 1904), it is proper to state that the manuscript of the present paper was in the hands of the editors before Dr. Converse's article appeared.

† Annals of Mathematics, ser. 2, vol. 3 (1901–1902), pp. 154–160.

$$l_1uv^2 + l_2uw^2 + m_1vw^2 + m_2vu^2 + n_1wu^2 + n_2wv^2 - 2k\,uvw = 0,$$

where the following relations subsist among the coefficients:

$$\begin{aligned} &l_1 \sin A \sin^2 B - l_2 \sin A \sin^2 C = m_1 \sin B \sin^2 C - m_2 \sin B \sin^2 A \\ &\quad = n_1 \sin C \sin^2 A - n_2 \sin C \sin^2 B = 2m \sin^2 A \sin^2 B \sin^2 C. \end{aligned} \tag{3}$$

Making use of the fact that the coefficients of u^3, v^3, and w^3 in (1) are zero, and employing the identical relation

$$(x + y + z)^3 - 27xyz = \lambda^3 + \mu^3 - 3\lambda\mu\nu,$$

it is easy to show that

$$p_1 = \tfrac{2}{3}\cos 3a, \quad p_2 = -\tfrac{2}{3}\cos(3C + 3a), \quad p_3 = -\tfrac{2}{3}\cos(3B - 3a);$$

and also that these values satisfy all the relations (3), except those involving m, whatever value a may have. From the last of these relations we find

$$m = \tan(C - B + 3a).$$

It is obvious from the form of equation (1) that the curve which it represents touches the sides of the triangle of reference; and, by varying the value of m, it is possible to make it represent any inscribed hypocycloid. Thus we may prove a number of theorems about a system of hypocycloids inscribed in a given triangle.

Locus of the Cusps. The tangential equations of the cusps are $x = 0$, $y = 0$, $z = 0$; hence the point-coordinates of one of the cusps, $x = 0$, are given by

$$\rho\xi = \frac{2}{3}\cos 3a + 2\cos a = \frac{8}{3}\cos^3 a,$$

$$\rho\eta = -\frac{2}{3}\cos(3C + 3a) - 2\cos(C + a) = -\frac{8}{3}\cos^3(C + a),$$

$$\rho\zeta = -\frac{2}{3}\cos(3B - 3a) - 2\cos(B - a) = -\frac{8}{3}\cos^3(B - a).$$

Eliminating ρ and a from these equations, we find for the locus of the cusps

$$a\sqrt[3]{\xi} + b\sqrt[3]{\eta} + c\sqrt[3]{\zeta} = 0,$$

or,

$$(a^3\xi + b^3\eta + c^3\zeta)^3 - 27a^3b^3c^3\xi\eta\zeta = 0.$$

This cubic curve has the sides of the triangle of reference as inflexional tangents, the straight line joining the points of inflexion being

$$a^3\xi + b^3\eta + c^3\zeta = 0.$$

Locus of the Centre. The tangential equation of the centre is $x + y + z = 0$; from which we easily obtain the point-equation of the locus of the centre in the form

$$\xi \sin 3A + \eta \sin 3B + \zeta \sin 3C = 0.$$

This is a straight line passing through the nine-point centre $[\cos(B - C), \cos(C - A), \cos(A - B)]$ of the triangle of reference. The nine-point centre is the centre of the hypocycloid enveloped by the asymptotes of the equilateral hyperbola (a result due to Steiner).

I have also shown, in a paper on the envelope of the axes of a system of conics passing through three fixed points,* that the envelope of the axes of the system of conics referred to above is also a pair of hypocycloids. The tangential equation of one of them is

$$\Sigma\left[u(v\cos C + w\cos B - u)\left\{u\cos(B - C) - v\cos B - w\cos C\right\}\right]$$
$$- \frac{2}{s+1}(v\cos C + w\cos B - u)(w\cos A + u\cos C - v)$$
$$(u\cos B + v\cos A - w) = 0,$$

where s is the secant of the angle between the asymptotes.

The same transformation may be applied to this equation; and by making use of the relations

$$[u^3]\cos C + [u^2v] = [v^3]\cos A + [v^2v], \text{ etc.},$$

where $[u^3]$ denotes the coefficient of u^3 in the equation, I find for the values of the constants:

$$3p_1 = \rho\cos A - 2\cos(B - C),$$
$$3p_2 = \rho\cos B - 2\cos(C - A),$$
$$3p_3 = \rho\cos C - 2\cos(A - B),$$
$$\rho = -4/s,\ 3a = B - C.$$

It may be shown that the hypocycloid represented by this equation touches the perpendicular bisectors of the sides of the triangle of reference;

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* *Transactions of the American Mathematical Society*, vol. 4 (1903), pp. 103–106.

and hence by varying s (or ρ) we have the equation of a system of hypocycloids touching three fixed *concurrent* straight lines.

The locus of the centre in this case is the straight line joining the circumcentre and the nine-point centre of the triangle of reference; and the centres of the two hypocycloids determined by one system of conics (of given eccentricity) are harmonically separated by these two points.

The locus of the cusps consists of three straight lines passing through the circumcentre: and the circle passing through the cusps touches two fixed straight lines passing through the circumcentre.

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ON THE INTEGRAL DIVISORS OF $a^n - b^n$

By Geo. D. Birkhoff and H. S. Vandiver*

The object of the following discussion is to develop some of the arithmetical properties of the divisors of $a^n - b^n$, and of forms related to it, in an elementary way. We have endeavored to give natural and concise proofs of the known theorems (I, II, III and IV) which lead up to the enunciation and proof of a new proposition (V). It is also shown that a principle in the theory of cyclotomic divisors may be used in giving a very simple proof of the irreducibility of $X^{p-1} + X^{p-2} + \cdots + 1$, p being a prime integer. All the numbers considered will be positive integers.

1. General Properties of the Cyclotomic Divisors. If a is greater than b and prime to it we write

$$V_n = a^n - b^n.$$

If $1, n', n'', n'''$, etc. are all the different divisors of n which are less than n, then V_n is divisible by V_1, $V_{n'}$, $V_{n''}$, $\cdots$, etc. A divisor of V_n which is prime to all these numbers is termed a *primitive divisor* of the form. Thus 5 is a primitive divisor of $2^4 - 1$. Other divisors are called *imprimitive*, and we may regard unity as a primitive divisor of any V.†

A characteristic property of primitive divisors is given by the following

Theorem I (Euler). *Any primitive divisor of V_n is of the form* $1 \pmod{n}$.

If unity is the only existing primitive divisor, the theorem holds. If others

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* The principal results in the present article were included in a communication made to the American Mathematical Society, at the meeting held Dec. 29, 1902. The authors are indebted to Professor E. S. Crawley and Dr. G. H. Hallett, of the University of Pennsylvania for several suggestions.

† In the classification of divisors of V_n the term *proper divisor* is employed by many writers, being defined as a factor which does not divide any of the forms $V_1, V_{n'}, V_{n''}$, etc. For examples, see Serret, *Cours d'algèbre supérieure* 5th edition, vol. 2, p. 145, and Dickson's *Linear Groups*, p. 20. According to this definition prime primitive divisors are prime proper divisors, and it is not difficult to see that the theory of proper divisors depends upon that of primitive divisors.

exist, let p be any prime one, and determine m from the congruence $a \equiv bm \pmod{p}$; then $V_n \equiv b^n m^m - b^n \equiv 0 \pmod{p}$ and since b is prime to p, $m^n - 1 \equiv 0 \pmod{p}$. It now follows from the definition of p, that m appertains to n, modulo p, so that $p - 1 \equiv 0 \pmod{n}$ or $p \equiv 1 \pmod{n}$, showing that any such prime divisor has the form $1 \pmod{n}$. Now any composite primitive divisor of V_n is the product of prime divisors of the same type. For if there is a divisor of the composite which is not primitive then the composite cannot be prime to V_1, $V_{n'}$, $V_{n''}$, etc., where $1, n', n''$, etc., have the same meaning as before. Hence all the primitive divisors are of the form $1 \pmod{n}$, and the theorem is proved.

Before proceeding with the further study of primitive divisors we must consider the two following theorems:

THEOREM II. *If V_1 is prime to n, then V_n / V_1 is prime to V_1.*

THEOREM III. *If p is prime and V_1/p^k is an integer prime to p, then V_p is divisible by p^{k+1}; it is not divisible by p^{k+2}, except when $V_1 = 2M$, where M is odd, in which case V_2 may be divisible by any power of 2 greater than 4.*

To prove Theorem II, we have

$$V_1 = a - b,$$

and $$a^n = (b + V_1)^n = b^n + nb^{n-1} V_1 + \frac{n(n-1)}{2!} b^{n-2} V_1^2 + \cdots,$$

or $$a^n - b^n = V_n = V_1\left(nb^{n-1} + \frac{n(n-1)}{2!} b^{n-2} V_1 + \cdots + V_1^{n-1}\right),$$

whence $$\frac{V_n}{V_1} \equiv nb^{n-1} \pmod{V_1}.$$

Now b is prime to V_1; therefore if n is prime to V_1, then V_n / V_1 is also.

To prove Theorem III put $V_1/p^k = M$, an integer prime to p, and suppose first, that p is an odd prime. Then

$$V_1 = a - b = Mp^k,$$

from which

$$\begin{aligned} a^p &= (b + Mp^k)^p \\ &= b^p + p^{k+1} b^{p-1} M + \cdots + p^{pk} M^p; \end{aligned}$$

or, since p is prime,

$$a^p - b^p = V_p = p^{k+1}\left(b^{p-1}M + \frac{p^k(p-1)}{2!}b^{p-2}M^2 + \cdots + p^{k(p-1)-1}M^p\right),$$

whence, p being odd,

$$\frac{V_p}{p^{k+1}} \equiv b^{p-1}M \pmod{p}.$$

It then follows that V_p/p^{k+1} is prime to p, since b and M are.

If we have $V_1/2^k = M_1$ say, where M_1 is odd and $k > 1$, then we find (as above) that V_2 is divisible by 2^{k+1} but not by 2^{k+2}. But in the case $k = 1$, we have

$$V_1 = a - b = 2M_1 \text{ and } V_2 = 4(bM_1 + M_1^2).$$

Then, since b and M are odd, V_2 is divisible by 8, or by a higher power of 2. This completes the proof of Theorem III.

We now resume the consideration of primitive divisors. Let the resolution of V_n into its distinct prime factors be

$$V_n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}.$$

If p_c, p_d, p_e, etc. are all the distinct prime primitive divisors of V_n, consider the product

$$P(n) = p_c^{a_c}p_d^{a_d}p_e^{a_e}\cdots$$

This number $P(n)$ shall be termed the *arithmetic primitive factor* of V_n, and is evidently divisible by any primitive divisor of V_n. Thus 5^2 or 25 is the arithmetical primitive factor of $7^4 - 1$, the primitive divisors being 1, 5 and 25. Similarly, unity is the arithmetical primitive factor of $2^6 - 1$.

The number $P(n)$ is closely related to a certain algebraic factor of V_n. We define this function by

$$F(n) = \frac{V_n \prod V_{\frac{n}{pq}} \prod V_{\frac{n}{pqrs}} \cdots}{\prod V_{\frac{n}{p}} \prod V_{\frac{n}{pqr}} \cdots} \qquad (1)$$

where p, q, r, s, etc. are the distinct prime divisors of n,

$$\prod V_{\frac{n}{p}} = V_{\frac{n}{p}} V_{\frac{n}{q}} V_{\frac{n}{r}} \cdots,$$

$$\prod V_{\frac{n}{pq}} = V_{\frac{n}{pq}} V_{\frac{n}{pr}} V_{\frac{n}{ps}} \cdots,$$

and similarly for the other products. In the numerator of (1), the indices of

the binomials are formed through the division of n by an even number of prime factors; in the denominator they are formed through the division by an odd number.

The form $F(n)$ has the following property:

$$V_n = F(1)F(n)F(n')F(n'')\cdots,$$

where 1, n, n', n'', etc. are all the distinct divisors of n. This function may also be defined by the equation

$$F(n) = (a - \omega_1 b)(a - \omega_2 b)\cdots(a - \omega_{\phi(n)} b)$$

($\omega_1, \omega_2, \cdots \omega_{\phi(n)}$, being the $\phi(n)$ primitive roots of $x^n - 1 = 0$), and can be conveniently termed the *algebraic primitive factor* of V_n, since it has no *algebraic* factor in common with V_1, $V_{n'}$, $V_{n''}$, etc.*

Let λ be a quantity defined by the relation

$$F(n) = \lambda P(n).$$

For the determination of λ we have,

THEOREM IV.† *If $n \neq 2$, then $\lambda = 1$, unless $P(n/p^a) \equiv 0 \pmod{p}$, n/p^a being an integer prime to p. In the latter case, $\lambda = p$.*

From the equation

$$V_n = F(1)\ F(n)\ F(n')\ F(n'')\cdots$$

we get

$$V_n / V_{\frac{n}{t}} \equiv 0 [\text{mod } F(n)],$$

where t is any divisor of n exclusive of unity. Taken in connection with Theorem II, this relation shows that $F(n)$ is prime to every imprimitive divisor of V_n which is prime to n, hence if we have $F(n) = \lambda P(n)$, λ is an integer each of whose prime factors must divide n. Suppose p is such a prime factor. From equation (1) we have immediately, writing $F(a,b,n)$ for the algebraic primitive factor of $a^n - b^n$,

$$F(a, b, n) = \frac{F(a^{p^a}, b^{p^a}, n')}{F(a^{p^{a-1}}, b^{p^{a-1}}, n')}$$

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* For the proofs of the foregoing relations see Bachmann's *Kreistheilung*, 1872, pp. 13–17. Notice that $\phi(n)$ means, as usual, the number of integers less than and prime to n.

† The first enunciation of this theorem, in its complete form, appears to be that of Sylvester, *American Journal Math.* vol. 2 (1879), p. 365. His demonstration is different from the one given here.

where $n = p^a n'$, and n' is prime to p. Applying the relation*

$$F(a^{p^k}, b^{p^k} n') \equiv [F(a, b, n')]^{p^k} \pmod{p},$$

we obtain

$$F(a, b, n) \equiv [F(a, b, n')]^{p^{a-1}(p-1)} \pmod{p}.$$

This shows that $F(a,b,n) \equiv 0 \pmod{p}$ only when $F(a,b,n') \equiv 0 \pmod{p}$. If the latter congruence is not satisfied for any p, then $F(n) = P(n)$. But if it is satisfied, then since $F(n')/P(n') = \lambda'$ is prime to p, p is also a divisor of $P(n')$, and is of the form 1 (mod n'). This gives $p > n/p^a$, hence p is uniquely determined as the largest prime dividing n, and we must have $\lambda = p^A$, where A is to be determined.

It can be shown that if $n > 2$, then $A = 1$. The only V in the denominator of the expression for $F(n)$ given in (1) which contains p is $V_{\frac{n}{p}}$, and in the numerator p divides V_n only, there being none other of the binomials in $F(n)$ having $F(a, b, n')$ as a factor. If $V_{\frac{n}{p}} = p^m K$, then $V_n = p^{m+1} K'$, where K and K' are prime to P, and $n > 2$ (Theorem III). Hence $F(n)$ is divisible by p but not by p^2 (for $n > 2$) and the theorem follows.

As a useful special case of theorem IV, we have :—

If p is prime, then $F(p) = \dfrac{a^p - b^p}{a - b}$, *and* $F(p) = pP(p)$ *or* $P(p)$ *according as $a - b$ is or is not congruent to zero, modulo p.*

The foregoing theorems enable us to demonstrate the following interesting proposition regarding the primitive divisors of V_n :—

THEOREM V. *If $n \neq 2$, then any form V_n possesses at least one primitive divisor* (other then unity), *with the single exception* $V_n = 2^6 - 1^6$.

To prove this, it is sufficient to show that $P(n) > 1$. We consider first the case in which $P(n) = \dfrac{F(n)}{p}$, where, as in Theorem IV, $n = p^a n'$, $p \equiv 1 \pmod{n'}$, and $F(n') \equiv 0 \pmod{p}$.

Since $F(n')$ is not zero, we have

$$F(n') \geqq p,$$

from the relation $F(n') \equiv 0 \pmod{p}$;

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* See Serret's *Cours d'algèbre supérieure*, 5th Edition, vol. 2, p. 134.

and therefore $$P(n) \geqq \frac{F(n)}{F(n')}.$$

Suppose, for the present, $n' \neq 1$.

Since a and b are integers, and $a > b$, we have

$$\begin{aligned} a^k - b^k &= (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}) \\ &\geqq a^{k-1} + a^{k-2}b + \cdots + b^{k-1}, \\ a^k - b^k &> a^{k-1}. \end{aligned}$$

Applying this inequality to the numerator of the expression for $F(n)$ in (1), we find

$$V_n \Pi V_{\frac{n}{p_i}} \Pi V_{\frac{n}{p_i p_j r_s}} \cdots > a^{n-1} \Pi a^{\frac{n}{p_i p_j}-1} \Pi a^{\frac{n}{p_i p_j r_s}-1} \cdots,$$

where $$\Pi a^{\frac{n}{p_i p_j}-1} = a^{\frac{n}{p_i p_j}-1} \cdot a^{\frac{n}{p_i p_r}-1} \cdot a^{\frac{n}{p_j p_r}-1} \cdots,$$

and similarly for the other products. Then,

$$\begin{aligned} V_n \Pi V_{\frac{n}{p_i}} \Pi V_{\frac{n}{p_i p_j r_s}} \cdots &> a^{n-1+\Sigma\left(\frac{n}{p_i p_j}-1\right)+\Sigma\left(\frac{n}{p_i p_j r_s}-1\right)\cdots}, \\ &> a^{-2^m-1+n+\Sigma\frac{n}{p_i}+\Sigma\frac{n}{p_i p_j}\cdots}, \end{aligned}$$

where m is the number of distinct prime factors in n.

For the denominator of $F(n)$ we have, since $a^k - b^k < a^k$,

$$\Pi V_{\frac{n}{p}} \Pi V_{\frac{n}{p_i r}} \cdots < a^{\Sigma\frac{n}{p}+\Sigma\frac{n}{p_i r}\cdots},$$

and by division of the inequalities for numerator and denominator of (1), and using the relation

$$\phi(n) = n - \Sigma\tfrac{n}{p} + \Sigma\tfrac{n}{p_i p_j} - \Sigma\tfrac{n}{p_i p_j r} + \cdots,$$

we obtain

$$F(n) > a^{\phi(n)-2^m-1}.$$

Now by applying the inequality $a^k - b^k < a^k$ to the numerator of $F(n')$ and $a^k - b^k > a^{k-1}$ to the denominator, we obtain

$$F(n') < a^{\phi(n')-2^{m-2}},$$

m, by hypothesis, not being unity. Hence

$$\frac{F(n)}{F(n')} > a^{\phi(n) - \phi(n') - 3 \cdot 2^{m-2}}.$$

Consider the expression $\phi(n) - \phi(n') - 3 \cdot 2^{m-2}$. Since $n' > 1$, we have $\phi(n') \geqq 2^{m-2}$. Also, $p^{a-1}(p-1) - 1 \geqq 3$, except when $p = 3$, noting that $p \neq 2$. Therefore

$$\phi(n')[p^{a-1}(p-1) - 1] \geqq 3 \cdot 2^{m-2},$$

or

$$\phi(n) - \phi(n') - 3 \cdot 2^{m-2} \geqq 0,$$

except when $p = 3$, that is, except when $n = 6$. Hence

$$\frac{F(n)}{F(n')} > 1, \text{ or } P(n) > 1,$$

except when $n = 6$. We have $F(6) = a^2 - ab + b^2$, which is greater than 3 except for $a = 2$, $b = 1$, in which case the arithmetical primitive factor is unity and no primitive divisor other than unity exists.

When $n' = 1$, then $n = p^a$ and from (1) $F(p^a) = \dfrac{V_n}{V_{\frac{n}{p}}}$, which expression is greater than p since the expanded forms contain p positive integral terms some of which are greater than unity. Hence $P(p^a) > 1$. The case $P(n) = F(n)$ remains to be disposed of. We have $F(n) > a^{\phi(n) - 2^{m-1}}$. Now obviously $\phi(n) \geqq 2^{m-1}$, hence $F(n) > 1$ and $P(n) > 1$. Theorem V is therefore completely proved. An immediate consequence of Theorem V is the following:—

Any form $a^n - b^n$ possesses at least one divisor of the form $1 \pmod{n}$, *other than unity.*

2. Applications of the Theory.

1. *There is an infinity of primes of the form* $1 \pmod{n}$.

Consider the infinite series

$$P(n),\ P(2n),\ P(3n),\ \cdots\ P(kn)\ \cdots.$$

The integers thus represented are all relatively prime and by Theorem V each contains at least one prime divisor of the form $1 \pmod{n}$. Consequently

the series furnishes an infinity of distinct primes of this form.*

2. *The function $F(p)$ is irreducible in the domain of real integers, p being prime.*†

Proof. Put $b = 1$, and suppose

$$F(p) = \frac{a^p - 1}{a - 1}$$

reducible; then

$$\frac{a^p - 1}{a - 1} = a^{p-1} + a^{p-2} + \cdots + 1 = f_1(a)\, f_2(a),$$

where $f_1(a)$ and $f_2(a)$ are polynomials in a, with integral coefficients.‡

In the above identity give a the values $0, 2, 3, 4, \cdots, p-1$. The special case of Theorem IV (already mentioned) gives for any of these values

$$F(p) = P(p),$$

and also $f_1(a) \equiv 1 \pmod{p}$, since it is a divisor of $P(p)$. Consequently the congruence $f_1(a) - 1 \equiv 0 \pmod{p}$ admits $p-1$ roots, which is impossible since p is prime, and the degree of $f_1(a)$ is less than $p-1$. $F(p)$ is therefore irreducible.

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*Evidently Theorem V is not necessary for the demonstration of this proposition, as a proof can be deduced from Theorem IV alone. In this latter form the proof has been given by a number of writers. For examples, see Kronecker, *Vorlesungen über Zahlentheorie*, pp. 440–441; Hilbert, *Algebraische Zahlkörper*, p. 334; Sylvester, *Comptes rendus*, vol. 104, pp. 1085–86.

†For other proofs of this theorem, see the references in H. J. Smith's *Report on the Theory of Numbers*, Works, vol. I, p. 127, footnote.

‡ *Cf.* a theorem of Gauss's; *Disquisitiones arithmeticae*, Art. 341.

TWO ELEMENTARY CONSTRUCTIONS IN COMPLEX TRIGONOMETRY

By A. E. Kennelly

It is well known, and is easily shown, that the coordinates x and y of any point of the rectangular hyperbola $x^2 - y^2 = 1$ are equal respectively to $\cosh(2A)$ and $\sinh(2A)$, where A is numerically equal to the area bounded by the axis of X, the hyperbola, and the radius-vector of the point (x,y); while the coordinates x and y of any point of the circle $x^2 + y^2 = 1$ are equal respectively to $\cos(2A)$ and $\sin(2A)$, where A is numerically equal to the area bounded by the axis of X, the circle, and the radius-vector of the point (x,y).

In figures 1 and 2, the shaded areas are each numerically equal to one half. In fig. 1, $Ob = \sin 1$ and $bB = \cos 1$. In fig. 2, $Ob = \sinh 1$ and $bB = \cosh 1$.

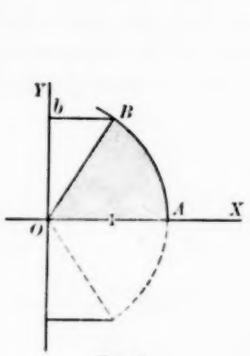

Fig. 1.

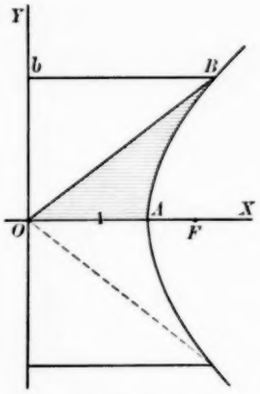

Fig. 2.

Assuming the familiar relations

$$\sin ja = j \sinh a,$$

$$\cos ja = \cosh a,$$

where $j = \sqrt{-1}$, we have

$$\sin(a + j\beta) = \sin a \cosh \beta + j \cos a \sinh \beta,$$

and

$$\cos(a + j\beta) = \cos a \cosh \beta - j \sin a \sinh \beta;$$

hence, $\sin(\alpha + j\beta)$, in the complex plane, is a point of intersection of the ellipse

$$\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = 1 \tag{1}$$

and the hyperbola

$$\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = 1, \tag{2}$$

while $\cos(\alpha + j\beta)$ is a point of intersection of the ellipse (1) and the hyperbola

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1; \tag{3}$$

the foci of all these curves being the points $(\pm 1, 0)$.

These facts suggest the following constructions:

1. Construction for $\sin(\alpha + j\beta)$. (Figure 3). Take $OA = 1$ along the negative end of the Y-axis. From OA as initial line, mark off the circular area $AOB = \alpha/2$. From OB as initial line, mark off the hyperbolic area $BOD = \beta/2$. Let C be the foot of the perpendicular from D

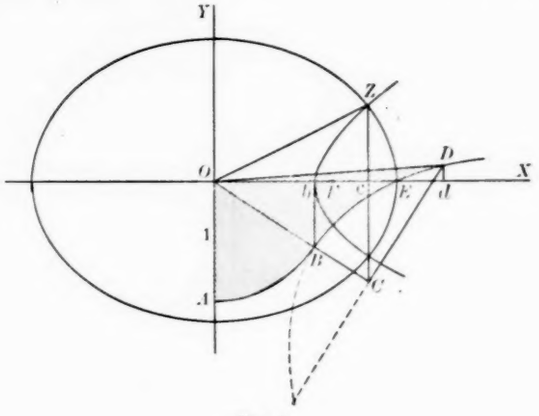

FIG. 3.

on OB produced. Drop perpendiculars from C and D on the axis of reals OX, at c and d respectively. About c as centre, rotate cd positively through $90°$ to cZ. Then will the complex vector $OZ = Oc + jcd$ be the required sine of the complex angle $\alpha + j\beta$.

In the case represented, $\sin(1 + j1) = 1.299 + j0.635 = 1.446 \text{ cis } 26° 3'$.

As β varies, Z moves along the hyperbola (2); as α varies, Z moves along the ellipse (1).

From the figure we get at once (if $x = Oc$ and $jy = cZ$),

$$\sin^{-1} OZ = \sin^{-1}(x \pm jy) = \sin^{-1} Ob \pm j \cosh^{-1} OE$$

$$= \sin^{-1}\left\{\frac{\sqrt{(1+x)^2 + y^2} - \sqrt{(1-x)^2 + y^2}}{2}\right\}$$

$$\pm j\cosh^{-1}\left\{\frac{\sqrt{(1+x)^2 + y^2} + \sqrt{(1-x)^2 + y^2}}{2}\right\},$$

since Ob is the semi-transverse axis of the hyperbola (2) through Z, while $OE = OC$ is the semi-major axis of the ellipse (1) through Z.

2. Construction for $\cos(\alpha + j\beta)$. (Figure 4.) Take $OA = 1$ along the positive end of the axis of reals. From OA as initial line mark off the circular area $AOB = \alpha/2$, etc., precisely as in the preceding paragraph. The complex vector $OZ = Oc + jcd$ thus obtained will be the required cosine of the complex angle $\alpha + j\beta$.

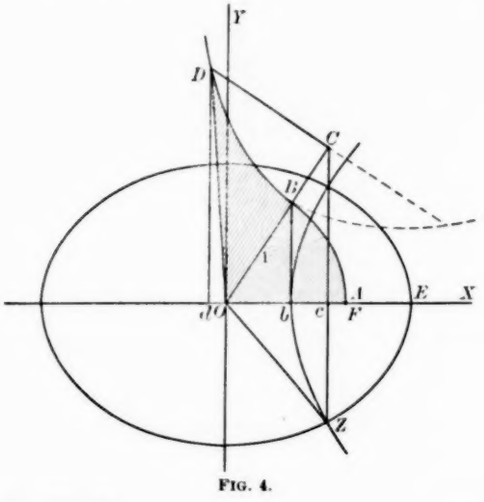

FIG. 4.

In the case represented,

$$\cos(1 + j1) = 0.834 - j\,0.989 = 1.293 \text{ cis } 49°52'.$$

As β varies, Z moves along the hyperbola (3); as α varies, Z moves along the ellipse (1).

From the figure,

$$\cos^{-1} OZ = \cos^{-1}(x \pm jy) = \cos^{-1} Ob \mp \cosh^{-1} OE$$

$$= \cos^{-1} \left\{ \frac{\sqrt{(1+x)^2 + y^2} - \sqrt{(1-x)^2 + y^2}}{2} \right\}$$

$$\mp j\cosh^{-1} \left\{ \frac{\sqrt{(1+x)^2 + y^2} + \sqrt{(1-x)^2 + y^2}}{2} \right\}.$$

It is evident that the same methods of construction serve also for $\sinh(\alpha + j\beta)$ and $\cosh(\alpha + j\beta)$. For, from $\sinh \theta = -j\sin(j\theta)$ and $\cosh\theta = \cos(j\theta)$,

we have, $\sinh(\alpha + j\beta) = j\sin(\beta - j\alpha)$,

and $\cosh(\alpha + j\beta) = \cos(\beta - j\alpha)$.

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NOTE ON STIRLING'S FORMULA

By S. A. Corey

Stirling's formula,* written in the notation ordinarily employed for Taylor's formula, is

$$f(a+x) = f(a) + \frac{x}{2}[f'(a+x) + f'(a)] - \frac{B_1 x^2}{2!}[f''(a+x) - f''(a)]$$
$$+ \frac{B_2 x^4}{4!}[f^{iv}(a+x) - f^{iv}(a)] - \cdots$$
$$+ (-1)^n \frac{B_n x^{2n}}{(2n)!}[f^{[2n]}(a+x) - f^{[2n]}(a)] + \cdots . \qquad (1)$$

Here B_1, B_2, $\cdots$ are Bernoulli's numbers, as follows: $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, $B_7 = 7/6$, etc.

If $f\left(a + \frac{x}{m}\right)$, $f\left[\left(a + \frac{2x}{m}\right) + \frac{x}{m}\right]$, $f\left[\left(a + \frac{3x}{m}\right) + \frac{x}{m}\right]$, $\cdots$, $f\left[\left(a + \frac{m-1}{m}x\right) + \frac{x}{m}\right]$

be developed successively by Stirling's formula, simplifying each by the aid of the preceding, the following formula is readily obtained:

$$f(a+x) = f(a) + \frac{x}{m \cdot 2}\left\{[f'(a+x) + f'(a)]\right.$$
$$\left. + 2\left[f'\left(a + \frac{x}{m}\right) + f'\left(a + \frac{2x}{m}\right) + f'\left(a + \frac{3x}{m}\right) + \cdots + f'\left(a + \frac{m-1}{m}x\right)\right]\right\}$$
$$- \frac{B_1 x^2}{m^2 \cdot 2!}[f''(a+x) - f''(a)] + \frac{B_2 x^4}{m^4 \cdot 4!}[f^{iv}(a+x) - f^{iv}(a)] - \cdots$$
$$+ (-1)^n \frac{B_n x^{2n}}{m^{2n} \cdot (2n)!}[f^{[2n]}(a+x) - f^{[2n]}(a)] + \cdots . \qquad (2)$$

This formula (2) is admirably adapted for the numerical computation of functions for which the value of the first derivative for each of the successive

* See Bertrand's *Calcul différentiel*, p. 332.

and equidistant steps, a, $a + x/m$, $a + 2x/m$, $a + 3x/m$, $\cdot\ \cdot\ \cdot$, $a + x$, is known or may be readily found, or of functions whose successive derivatives increase rapidly in complexity with the order of the derivatives. The special value of the formula will be understood from the following considerations:—

1. No odd-numbered derivatives, except the first derivative, enter into the formula and their computation is therefore not required, except in so far as may be necessary to determine the form of the even-numbered derivatives of higher orders.

2. The values of the even-numbered derivatives need be found for only two values of the variable, viz., $x = 0$ and $x = x$, the values of such derivatives for the intermediate steps having been entirely eliminated from the formula.

3. The value of the general term may be made as small as desired by making m large enough, that is by finding the value of the first derivative for a sufficiently large number of intermediate points, corresponding to equidistant values of x.

As (2) is but the sum of a finite number of series developed by Stirling's formula, it follows that (2) is convergent if each of the component series is convergent, and, consequently, that the same test which determines the convergence of each of the component series determines the convergence of (2).

The following examples will serve to illustrate the usefulness of formula (2) for numerical computation. In each example the value of m and the highest value of n used are given, and the results obtained are correct to the last place of decimals.

(1) $$\int_0^1 \frac{dx}{1+x} = 0.6931471805 \quad (m = 10,\ n = 3),$$

(2) $$\int_0^{3\pi} \frac{\sin x}{x}\, dx = 1.67476 \qquad (m = 10,\ n = 2),$$

(3) $$\int_0^{\frac{\pi}{4}} \tan x\, dx = 0.346573 \qquad (m = 10,\ n = 1),$$

(4) $$\int_0^{\frac{\pi}{4}} \left(\frac{2 - x^2}{2 - 2x^2}\right)^{\frac{1}{2}} dx = 0.84829 \qquad (m = 10,\ n = 1).$$

HITEMAN, IOWA, OCTOBER 1903.

NOTE ON SYLOW'S THEOREM

By G. A. Miller

Sylow's theorem is of such fundamental importance that any improvement in stating it is of interest. We desire to suggest the following form:

Theorem. *If the order of a group (G) is $p^\alpha m$, p being any prime which does not divide m, then G contains at least one subgroup (P_α) of order p^α. Moreover, if P_α contains only one subgroup (P_β) of a given type, then all the subgroups of this type in G are conjugate and their number is of the form $1 + kp$. The order of G may be written in the form $p^\beta n(1 + kp)$, $p^\beta n$ being the order of the largest subgroup which transforms P_β into itself.*

The first sentence of this theorem does not differ materially from the one usually given. The rest is more general than the usual statement. It reduces to this when β has the special value α. It gives more information about G than the usual form does only when something is known about P_α. For example, when P_α is cyclic, it follows that all the subgroups whose order is a given power of p form a single conjugate set and that the number of these subgroups is $1 + kp$. The additional information conveyed in these cases seems to justify the slight increase in the length of the theorem.

If we assume Sylow's theorem (in its usual form) proved, the more general statement given above follows very easily. Such a subgroup of order p^β could not transform another subgroup of this type into itself, otherwise P_α would include more than one such subgroup of order p^β. From this it follows exactly as in the case for p^α that the number of these subgroups of order p^β is of the form $1 + kp$ and that they form a single conjugate set.

Stanford University,
October, 1903.

THE CONDITIONS FOR A PLAIT POINT

By Paul Saurel

If a plane, tangent to a surface in two points, moves in such a way that the two points of contact tend to coalesce, the limiting position of these points of contact is called a plait point.* This definition applies not only to a surface in space of three dimensions but also to a hypersurface in a space of n dimensions.

Let us suppose that the equation of a surface in a space of $n + 1$ dimensions is

$$x = f(x_1, x_2, \cdots, x_n), \tag{1}$$

and let us write

$$\Delta = \begin{vmatrix} f_{11}, f_{12}, \cdots, f_{1n} \\ f_{21}, f_{22}, \cdots, f_{2n} \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ f_{n1}, f_{n2}, \cdots, f_{nn} \end{vmatrix}, \tag{2}$$

in which

$$f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{3}$$

At a plait point of the surface, every determinant of order n that can be formed from the matrix

$$\begin{matrix} f_{11}, & f_{12}, & \cdots, & f_{1n} \\ f_{21}, & f_{22}, & \cdots, & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1}, & f_{n2}, & \cdots, & f_{nn} \\ \dfrac{\partial \Delta}{\partial x_1}, & \dfrac{\partial \Delta}{\partial x_2}, & \cdots, & \dfrac{\partial \Delta}{\partial x_n} \end{matrix} \tag{4}$$

reduces to zero.

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* Korteweg, *Archives Néerlandaises des sciences exactes et naturelles*, vol. 24, p. 57 (1890).

These conditions are due to Gibbs.* The reasoning by which he establishes them is, however, stated in physical terms. It may therefore be well to divest his argument of its physical garb and to present it, or rather an equivalent of it, in mathematical terms.

The equations which express the coincidence of the tangent planes drawn to the surface 1 at two points x, x_1, x_2, $\cdots$, x_n and x', x_1', x_2', $\cdots$, x_n' may be written

$$f_i' = f_i \qquad i = 1, 2, \cdots, n, \tag{5}$$

$$f' - \sum_{i=1}^{n} x_i' f_i' = f - \sum_{i=1}^{n} x_i f_i, \tag{6}$$

in which

$$f_i = \frac{\partial f}{\partial x_i}, \tag{7}$$

and the primed letters indicate quantities connected with the point x', x_1', x_2', $\cdots$, x_n'.

If we suppose the two points of contact to be very near together, these equations may be written

$$\sum_{j=1}^{n} (x_j' - x_j) f_{ij} + \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j' - x_j)(x_k' - x_k) f_{ijk} + \cdots = 0, \qquad i = 1, 2, \cdots, n, \tag{8}$$

$$\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i' - x_i)(x_j' - x_j) f_{ij} + \frac{1}{2.3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_i' - x_i)(x_j' - x_j)(x_k' - x_k) f_{ijk} + \cdots = 0, \tag{9}$$

in which

$$f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}. \tag{10}$$

The omitted terms are of the third and fourth orders respectively.

Multiplying equations 8 by $x_1' - x_1$, $x_2' - x_2$, $\cdots$, $x_n' - x_n$ and subtracting one half their sum from equation 9, we find that this equation can be replaced by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_i' - x_i)(x_j' - x_j)(x_k' - x_k) f_{ijk} + \cdots = 0. \tag{11}$$

* On the Equilibrium of Heterogeneous Substances, *Transactions of the Connecticut Academy of Arts and Sciences*, vol. 3, part 1, p. 191 (1876).

The conditions for a plait point are thus, first, that the equations

$$\sum_{j=1}^{n} (x_j' - x_j) f_{ij} = 0, \qquad i = 1, 2, \cdots, n, \tag{12}$$

should be compatible, which requires that

$$\Delta = 0;$$

and, second, that the values of $x_j' - x_j$ which satisfy equations 12 should also satisfy the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_i' - x_i)(x_j' - x_j)(x_k' - x_k) f_{ijk} = 0. \tag{14}$$

If we denote the minor of Δ which corresponds to f_{ij} by F_{ij}, the values of $x_1' - x_1, x_2' - x_2, \cdots, x_n' - x_n$ which satisfy equations 12 are given by any one of the n sets of equations

$$x_1' - x_1 : x_2' - x_2 : \cdots : x_n' - x_n = F_{r1} : F_{r2} : \cdots : F_{rn}, \qquad r = 1, 2, \cdots, n. \tag{15}$$

Now, equation 14 can be written in the form

$$\sum_{k=1}^{n} (x_k' - x_k) \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i' - x_i)(x_j' - x_j) f_{ijk} = 0. \tag{16}$$

If we compare this equation with equations 12 it follows that every determinant of order n formed from the matrix

$$\begin{array}{cccc} f_{11}, & f_{12}, & \cdots, & f_{1n} \\ f_{21}, & f_{22}, & \cdots, & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1}, & f_{n2}, & \cdots, & f_{nn} \\ A_1, & A_2, & \cdots, & A_n, \end{array} \tag{17}$$

where $$A_k = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i' - x_i)(x_j' - x_j) f_{ijk},$$

must be equal to zero.

We shall now show that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (x_i' - x_i)(x_j' - x_j) f_{ijr} \tag{18}$$

is proportional to

$$\frac{\partial \Delta}{\partial x_r},$$

so that the matrix 17 is equivalent to the matrix 4. We shall thus have established Gibbs's conditions.

From equations 15 it follows that 18 is proportional to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ijr}\, F_{si}\, F_{sj}\,. \tag{19}$$

On the other hand

$$\frac{\partial \Delta}{\partial x_r} = \begin{vmatrix} f_{11r}, f_{12r}, \cdots, f_{1nr} \\ f_{21}, f_{22}, \cdots, f_{2n} \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ f_{n1}, f_{n2}, \cdots, f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11}, f_{12}, \cdots, f_{1n} \\ f_{21r}, f_{22r}, \cdots, f_{2nr} \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ f_{n1}, f_{n2}, \cdots, f_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} f_{11}, f_{12}, \cdots, f_{1n} \\ f_{21}, f_{22}, \cdots, f_{2n} \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ \cdot \quad \cdot \quad\quad \cdot \\ f_{n1r}, f_{n2r}, \cdots, f_{nnr} \end{vmatrix}$$

$$\begin{aligned} &= f_{11r}\, F_{11} + f_{12r}\, F_{12} + \cdots + f_{1nr}\, F_{1n} \\ &+ f_{21r}\, F_{21} + f_{22r}\, F_{22} + \cdots + f_{2nr}\, F_{2n} \\ &\quad \cdot \quad\quad \cdot \quad\quad \cdot \quad\quad \cdot \quad\quad \cdot \\ &+ f_{n1r}\, F_{n1} + f_{n2r}\, F_{n2} + \cdots + f_{nnr}\, F_{nn} \end{aligned}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ijr}\, F_{ij}. \tag{20}$$

From this equation we get

$$F_{ss}\,\frac{\partial \Delta}{\partial x_r} = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ijr}\, F_{ss}\, F_{ij}. \tag{21}$$

Now since $\Delta = 0$, it follows from an elementary theorem in determinants that

$$F_{ss}\, F_{ij} - F_{si}\, F_{sj} = 0. \tag{22}$$

A comparison of 18, 19, 21, and 22 then shows immediately that expression 18 is proportional to $\dfrac{\partial \Delta}{\partial x_r}$.

The theorem is thus established.

The above conclusion fails if $F_{ss} = 0$ for every value of s. But, in that case, since

$$F_{ss}\, F_{rr} = F_{rs}^2,$$

it follows that, for all values of i and j, $F_{ij} = 0$. The equations 12 no longer

uniquely determine the ratios of $x_1' - x_1$, $x_2' - x_2$, $\cdots$, $x_n' - x_n$ and the singularity is of a more complicated character.

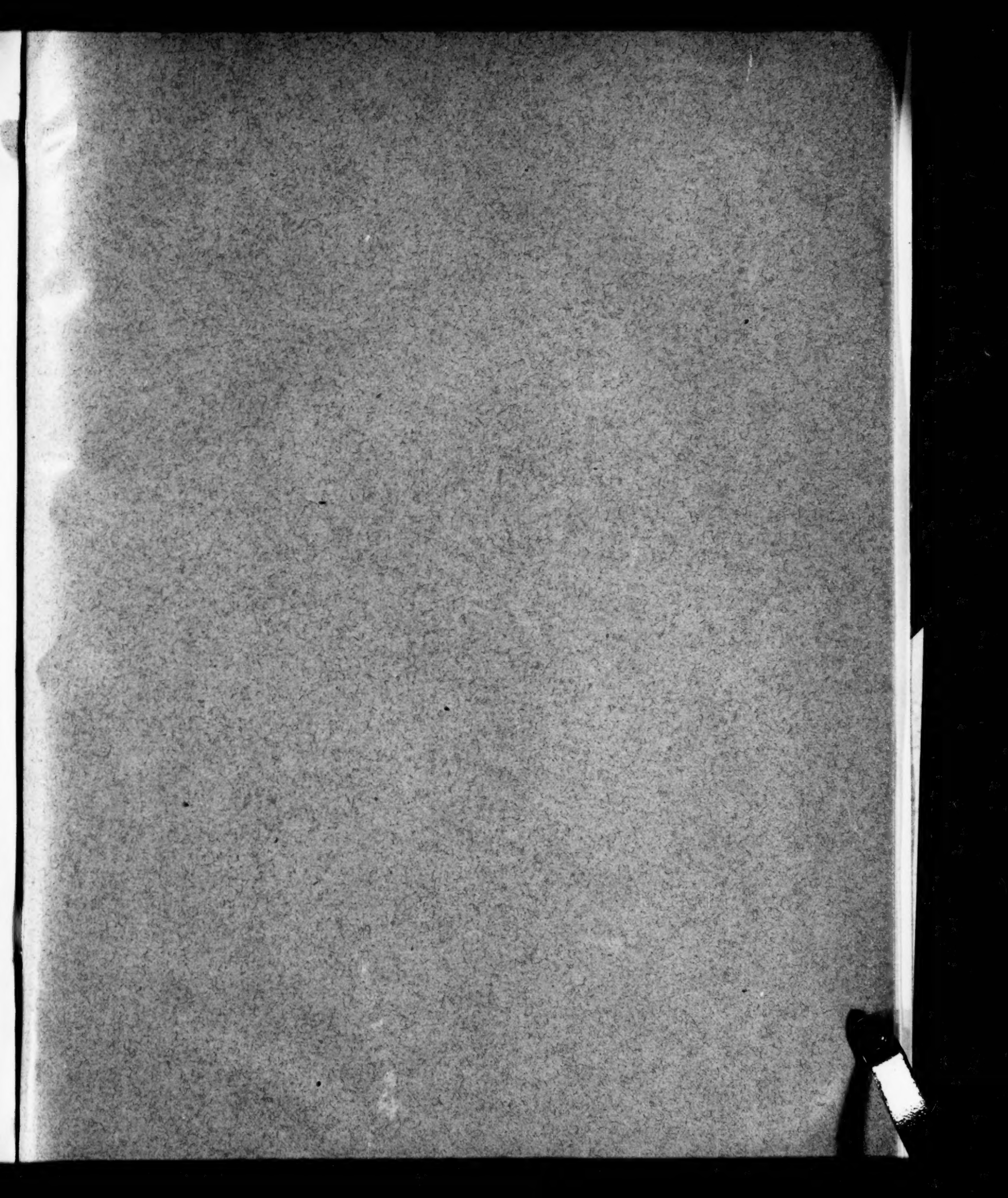
NEW YORK, NOVEMBER 26, 1903.

ACKNOWLEDGMENT.

The demonstration given in my note "On a Theorem of Kinematics" in the ANNALS OF MATHEMATICS, second series, vol. 2, p. 159, is, in substance, the same as that given by Kelvin and Tait in their *Treatise on Natural Philosophy*, vol. 1, p. 78. I regret that I did not become aware of the fact sooner.

PAUL SAUREL.

NOVEMBER, 1903.



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